

(A_σ) -double sequence spaces defined by Orlicz function and double statistical convergence

Ekrem Savaş

Istanbul Ticaret University, Department of Mathematics, Uskudar, Istanbul, Turkey

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Abstract

The purpose of this paper is to introduce and study an idea of strong double (A_σ) -convergence sequences with respect to an Orlicz function. In addition, we define the double (A_σ) -statistical convergence and establish some connections between the spaces of strong double (A_σ) -convergence sequences and the space of double (A_σ) -statistical convergence.
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1. Introduction and background

Let l_∞ be the Banach space of bounded $x = (x_k)$ with the usual norm $\|x\| = \sup_n |x_n|$. A sequence $x \in l_\infty$ is said to be almost convergent if all of its Banach limits coincide. Let \hat{c} denote the space of all almost convergent sequences. Lorentz [1] proved that

$$\hat{c} = \{x \in l_\infty : \lim_m t_{m,n}(x) \text{ exists uniformly in } n\}$$

where

$$t_{m,n}(x) = \frac{x_n + x_{n+1} + \cdots + x_{m+n}}{m+1}.$$

The following space of strongly almost convergent sequence was introduced by Maddox in [2]

$$[\hat{c}] = \{x \in l_\infty : \lim_m t_{m,n}(|x - Le|) \text{ exists uniformly in } n \text{ for some } L\}$$

where $e = (1, 1, \dots)$.

Let σ be a one-to-one mapping from the set of natural numbers into itself. A continuous linear functional ϕ on l_∞ is said to be an invariant mean or a σ -mean provided that:

- i. $\phi(x) \geq 0$ when the sequence $x = (x_k)$ is such that $x_k \geq 0$ for all k ,

E-mail address: ekremsavas@yahoo.com.

- ii. $\phi(e) = 1$ where $e = (1, 1, 1, \dots)$, and
- iii. $\phi(x) = \phi(x_{\sigma(k)})$ for all $x \in l_\infty$.

For certain class of mapping σ every invariant mean φ extends the limit functional on space c , in the sense that $\varphi(x) = \lim x$ for all $x \in c$. The space $[V_\sigma]$ is of strongly σ -convergent sequence was introduced by Mursaleen [3] as follows: A sequence $x = (x_k)$ is said to be strongly σ -convergent if there exists a number L such that

$$\frac{1}{k} \sum_{i=1}^k |x_{\sigma^i(m)} - L| \rightarrow 0 \quad (1.1)$$

as $k \rightarrow \infty$ uniformly in m . We will denote $[V_\sigma]$ as the set of all strongly σ -convergent sequences. When (1.1) holds we write $[V_\sigma] - \lim x = L$. If we let $\sigma(m) = m + 1$, then $[V_\sigma] = [\hat{c}]$, which is defined by Maddox in [2]. Thus strong σ -convergence generalizes the concept of strong almost convergence sequence space.

Recall in [4] that an Orlicz function $M : [0, \infty) \rightarrow [0, \infty)$ is continuous, convex, nondecreasing function such that $M(0) = 0$ and $M(x) > 0$ for $x > 0$, and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Subsequently Orlicz function was used to define sequence spaces by Parashar and Choudhary [5] and others.

If convexity of Orlicz function M is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called Modulus function, which was presented and discussed by Ruckle [6] and Maddox [7]. Let s'' denote the set of all double sequences of real numbers. Before proceeding further let us recall a few concepts, which we shall use throughout this paper.

Definition 1.1. Let A denote a four-dimensional summability method that maps the complex double sequences x into the double sequence Ax where the mn -th term to Ax is as follows:

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

By a bounded double sequence we shall mean there exists a positive number K such that $|x_{k,l}| < K$ for all (k, l) , and denote such bounded by

$$\|x\|_{(\infty,2)} = \sup_{k,l} |x_{k,l}| < \infty.$$

We shall also denote the set of all bounded double sequences by l_∞'' . We also note in contrast to the case for single sequence, a P -convergent double sequence need not be bounded. In [8], Pringsheim presented the following definition:

Definition 1.2. A double sequence $x = (x_{k,l})$ has a Pringsheim limit L (denoted by $P\text{-}\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We shall describe such an x more briefly as “ P -convergent”.

Following Hardy’s work Robison in 1926 presented a four dimensional analogue of regularity for double sequences in which he added an additional assumption of boundedness. This assumption was made because a double sequence which is P -convergent is not necessarily bounded. Along these same lines, Robison and Hamilton presented a Silverman–Toeplitz type multidimensional characterization of regularity in [9,10]. The definition of the regularity for four dimensional matrices will be stated next, followed by the Robison–Hamilton characterization of the regularity of four- dimensional matrices.

Definition 1.3. The four dimensional matrix A is said to be RH-regular if it maps every bounded P -convergent sequence into a P -convergent sequence with the same P -limit.

Theorem 1.1. The four-dimensional matrix A is RH-regular if and only if

- RH_1 : $P\text{-}\lim_{m,n} a_{m,n,k,l} = 0$ for each k and l ;
- RH_2 : $P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} = 1$;
- RH_3 : $P\text{-}\lim_{m,n} \sum_{k=1}^{\infty} |a_{m,n,k,l}| = 0$ for each l ;
- RH_4 : $P\text{-}\lim_{m,n} \sum_{l=1}^{\infty} |a_{m,n,k,l}| = 0$ for each k ;

RH_5 : $\sum_{k,l=1,1}^{\infty,\infty} |a_{m,n,k,l}|$ is P -convergent; and
 RH_6 : there exist positive numbers A and B such that

$$\sum_{k,l>B} |a_{m,n,k,l}| < A.$$

The class of sequences which are strongly Cesaro summable with respect to an Orlicz function was introduced and studied in [5]. In this paper we introduce and study the concept of strong double A_σ -summable with respect to an Orlicz function and also some properties of this sequence space is examined. Before we can state our main results, first we shall present the following definition by combining a four-dimensional matrix transformation A and Orlicz function.

2. Main results

Definition 2.1. Let M be an Orlicz function and $A = (a_{m,n,k,l})$ be a nonnegative RH-regular summability matrix method, and $(p_{k,l})$ be any factorable double sequence of strictly positive real numbers. We now present the following double sequence spaces:

$$\begin{aligned} \omega''_0(A_\sigma, M, p) &= \left\{ x \in s'' : P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)}|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ &\quad \left. \text{uniformly in } (p, q), \text{ for some } \rho > 0 \right\}, \\ \omega''(A_\sigma, M, p) &= \left\{ x \in s'' : P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ &\quad \left. \text{uniformly in } (p, q), \text{ for some } \rho > 0, \text{ some } L \right\}, \end{aligned}$$

and

$$\omega''_\infty(A_\sigma, M, p) = \left\{ x \in s'' : \sup_{m,n,p,q} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)}|}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}$$

where $\sigma^{k,l}(p, q)$ is a one to one mapping from $N \times N$ into itself, (N is the set of the natural numbers).

Let us consider a few special cases of the above definition.

(1) In particular when $\sigma(p, q) = (p + 1, q + 1)$ we have

$$\begin{aligned} \omega''_0(\hat{A}, M, p) &= \left\{ x \in s'' : P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{k+p,l+q}|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ &\quad \left. \text{uniformly in } (p, q), \text{ for some } \rho > 0 \right\}, \\ \omega''(\hat{A}, M, p) &= \left\{ x \in s'' : P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{k+p,l+q} - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ &\quad \left. \text{uniformly in } (p, q), \text{ for some } \rho > 0, \text{ some } L \right\}, \end{aligned}$$

and

$$\omega''_\infty(\hat{A}, M, p) = \left\{ x \in s'' : \sup_{m,n,p,q} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{k+p,l+q}|}{\rho} \right) \right]^{p_{k,l}} < \infty, \right\}.$$

(2) If we take $M(x) = x$ and $p_{k,l} = 1$ for all (k, l) then we have

$$\omega_0''(A_\sigma) = \left\{ x \in s'' : P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} |x_{\sigma^{k,l}(p,q)}| = 0, \text{ uniformly in } (p, q), \right\},$$

$$\omega''(A_\sigma) = \left\{ x \in s'' : P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} |x_{\sigma^{k,l}(p,q)} - L| = 0, \text{ uniformly in } (p, q), \text{ for some } L \right\},$$

and

$$\omega_\infty''(A_\sigma) = \left\{ x \in s'' : \sup_{m,n,p,q} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} |x_{\sigma^{k,l}(p,q)}| < \infty \right\}.$$

(3) If we take $A = (C, 1, 1)$, which is double Cesàro matrix, we have (see, [11])

$$(\omega_\sigma'', M, p)^0 = \left\{ x \in s'' : P\text{-}\lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)}|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ \left. \text{uniformly in } (p, q), \text{ for some } \rho > 0 \right\},$$

$$(\omega_\sigma'', M, p) = \left\{ x \in s'' : P\text{-}\lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ \left. \text{uniformly in } (p, q), \text{ for some } \rho > 0, \text{ some } L \right\},$$

and

$$(\omega_\sigma'', M, p)^\infty = \left\{ x \in s'' : \sup_{m,n,p,q} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)}|}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}.$$

(4) Let us consider the following notations and definition. The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing integers sequences such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \rightarrow \infty \quad \text{as } r \rightarrow \infty,$$

$$l_0 = 0, \quad h_s = l_s - l_{s-1} \rightarrow \infty \quad \text{as } s \rightarrow \infty,$$

and let $h_{r,s} = h_r h_s$, $\theta_{r,s}$ is determine by $I_{r,s} = \{(i, j) : k_{r-1} < i \leq k_r \text{ \& } l_{s-1} < j \leq l_s\}$. If we take

$$a_{r,s,k,l} = \begin{cases} \frac{1}{h_{r,s}}, & \text{if } (k, l) \in I_{r,s}; \\ 0 & \text{otherwise.} \end{cases}$$

We are granted (see, [13])

$$(\omega_\sigma'', \theta, M, p)^0 = \left\{ x \in s'' : P\text{-}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)}|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ \left. \text{uniformly in } (p, q), \text{ for some } \rho > 0 \right\},$$

$$(\omega_\sigma'', \theta, M, p) = \left\{ x \in s'' : P\text{-}\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ \left. \text{uniformly in } (p, q), \text{ for some } \rho > 0, \text{ some } L \right\},$$

and

$$(\omega''_{\sigma}, \theta, M, p)^{\infty} = \left\{ x \in s'' : \sup_{r,s,p,q} \frac{1}{\bar{h}_{r,s}} \sum_{(k,l) \in I_{r,s}} \left[M \left(\frac{|x_{\sigma^{k,l}}(p,q)|}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}.$$

(5) As a final illustration let

$$a_{i,j,k,l} = \begin{cases} \frac{1}{\bar{\lambda}_{i,j}}, & \text{if } k \in I_i = [i - \lambda_i + 1, i] \text{ and } l \in I_j = [j - \lambda_j + 1, j] \\ 0, & \text{otherwise} \end{cases}$$

where $\bar{\lambda}_{i,j}$ by $\lambda_i \mu_j$. Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ be two non-decreasing sequences of positive real numbers such that each tending to ∞ and $\lambda_{i+1} \leq \lambda_i + 1$, $\lambda_1 = 0$ and $\mu_{j+1} \leq \mu_j + 1$, $\mu_1 = 0$. Then our definitions reduce to the following (see, [12])

$$\begin{aligned} (\omega''_{\sigma}, \bar{\lambda}, M, p)^0 &= \left\{ x \in s'' : P\text{-}\lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} \left[M \left(\frac{|x_{\sigma^{k,l}}(p,q)|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ &\quad \left. \text{uniformly in } (p, q), \text{ for some } \rho > 0 \right\}, \\ (\omega''_{\sigma}, \bar{\lambda}, M, p) &= \left\{ x \in s'' : P\text{-}\lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} \left[M \left(\frac{|x_{\sigma^{k,l}}(p,q) - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \right. \\ &\quad \left. \text{uniformly in } (p, q), \text{ for some } \rho > 0, \text{ some } L \right\}, \end{aligned}$$

and

$$(\omega''_{\sigma}, \bar{\lambda}, M, p)^{\infty} = \left\{ x \in s'' : \sup_{i,j,p,q} \frac{1}{\bar{\lambda}_{i,j}} \sum_{k \in I_i, l \in I_j} \left[M \left(\frac{|x_{\sigma^{k,l}}(p,q)|}{\rho} \right) \right]^{p_{k,l}} < \infty \right\}.$$

We now prove

Theorem 2.1. Let $p = p_{k,l}$ be bounded. Then $\omega''_0(A_{\sigma}, M, p)$, $\omega''(A_{\sigma}, M, p)$, and $\omega''_{\infty}(A_{\sigma}, M, p)$ are linear spaces over the set of complex numbers \mathbb{C} .

Proof. We consider only $\omega''_0(A_{\sigma}, M, p)$. The others can be treated similarly. Let $x, y \in \omega''_0(A_{\sigma}, M, p)$ and both α and β complex numbers. Since x and y are in $\omega''_0(A_{\sigma}, M, p)$ there exist some positive ρ_1 and ρ_2 such that

$$P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M \left(\frac{|x_{\sigma^{k,l}}(p,q)|}{\rho_1} \right)^{p_{k,l}} = 0$$

and

$$P\text{-}\lim_{m,n} \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} M \left(\frac{|x_{\sigma^{k,l}}(p,q)|}{\rho_2} \right)^{p_{k,l}} = 0$$

uniformly in (p, q) . Write $\rho_3 = \max\{2|\alpha|\rho_1, 2|\beta|\rho_2\}$. Since M is a nondecreasing convex function we obtain the following inequality:

$$\begin{aligned} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M \left(\frac{|\alpha x_{\sigma^{k,l}}(p,q) + \beta y_{\sigma^{k,l}}(p,q)|}{\rho_3} \right)^{p_{k,l}} &\leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} M \left(\frac{|\alpha x_{\sigma^{k,l}}(p,q)|}{\rho_3} + \frac{|\beta y_{\sigma^{k,l}}(p,q)|}{\rho_3} \right)^{p_{k,l}} \\ &\leq \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \frac{1}{2^{p_{k,l}}} \left[M \left(\frac{|x_{\sigma^{k,l}}(p,q)|}{\rho_1} \right) + M \left(\frac{|y_{\sigma^{k,l}}(p,q)|}{\rho_2} \right) \right]^{p_{k,l}} \end{aligned}$$

$$\begin{aligned} &\leq \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}}(p,q)|}{\rho_1} \right) + M \left(\frac{|y_{\sigma^{k,l}}(p,q)|}{\rho_2} \right) \right]^{p_{k,l}} \\ &\leq C \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}}(p,q)|}{\rho_1} \right) \right]^{p_{k,l}} + C \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|y_{\sigma^{k,l}}(p,q)|}{\rho_2} \right) \right]^{p_{k,l}} \end{aligned}$$

$\rightarrow 0$ as $m, n \rightarrow \infty$ in Pringsheim sense uniformly in (p, q) , where $\sup p_{k,l} = H$ and $C = \max\{1, 2^{H-1}\}$. Thus $\omega''_0(A_\sigma, M, p)$ is a linear space. \square

The above proof can easy be modified to prove the following theorem:

Theorem 2.2. If M be an Orlicz function then $\omega''_0(A_\sigma, M, p) \subset \omega''(A_\sigma, M, p) \subset \omega''_\infty(A_\sigma, M, p)$.

Definition 2.2. An Orlicz function M is said to satisfy Δ_2 -condition for all values of u , if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$ for all $u \geq 0$. The Δ_2 -condition is equivalent to the satisfaction of the following inequality $M(lu) \leq K(l)M(u)$ for all values of u and for $l \geq 1$.

Theorem 2.3. Let A be a nonnegative RH-regular summability matrix method and M be a Orlicz function which satisfies the Δ_2 -condition. Then $\omega''_0(A_\sigma, p) \subset \omega''_0(A_\sigma, M, p)$, $\omega''(A_\sigma, p) \subset \omega''(A_\sigma, M, p)$, and $\omega''_\infty(A_\sigma, p) \subset \omega''_\infty(A_\sigma, M, p)$.

Proof. Let $x \in \omega''(A_\sigma, p)$, then

$$s_{m,n}^{p,q} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} |x_{\sigma^{k,l}}(p,q) - L|^{p_{k,l}} \rightarrow 0 \quad (2.1)$$

as $m, n \rightarrow \infty$ uniformly in (p, q) in the Pringsheim sense. Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \frac{\epsilon}{2}$ for $0 \leq t \leq \delta$. Write $y_{\sigma^{k,l}}(p,q) = |x_{\sigma^{k,l}}(p,q) - L|$ and consider

$$\begin{aligned} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} [M(y_{\sigma^{k,l}}(p,q))]^{p_{k,l}} &= \sum_{k,l=0,0; y_{\sigma^{k,l}}(p,q) \leq \delta}^{\infty,\infty} a_{m,n,k,l} [M(y_{\sigma^{k,l}}(p,q))]^{p_{k,l}} \\ &+ \sum_{k,l=0,0; y_{\sigma^{k,l}}(p,q) > \delta}^{\infty,\infty} a_{m,n,k,l} [M(y_{\sigma^{k,l}}(p,q))]^{p_{k,l}}. \end{aligned}$$

Since M is continuous we obtain

$$\sum_{k,l=0,0; y_{\sigma^{k,l}}(p,q) \leq \delta}^{\infty,\infty} a_{m,n,k,l} [M(y_{\sigma^{k,l}}(p,q))]^{p_{k,l}} \leq \epsilon^H \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}$$

and for $y_{\sigma^{k,l}}(p,q) > \delta$ we have the fact that

$$y_{\sigma^{k,l}}(p,q) < \frac{y_{\sigma^{k,l}}(p,q)}{\delta} < \left[1 + \frac{y_{\sigma^{k,l}}(p,q)}{\delta} \right]$$

where $[t]$ denotes the integer part of t and since M is nondecreasing and convex we have

$$M(y_{\sigma^{k,l}}(p,q)) < M \left(\left[1 + \frac{y_{\sigma^{k,l}}(p,q)}{\delta} \right] \right) < \frac{M(2)}{2} + \frac{1}{2} M \left(\frac{2y_{\sigma^{k,l}}(p,q)}{\delta} \right).$$

Since M satisfies the Δ_2 -condition, therefore there exists $K \geq 1$ such that

$$M(y_{\sigma^{k,l}}(p,q)) < \frac{Ky_{\sigma^{k,l}}(p,q)}{2\rho} M(2) + \frac{Ky_{\sigma^{k,l}}(p,q)}{2\rho} M(2) < K \frac{y_{\sigma^{k,l}}(p,q)}{\rho} M(2).$$

Hence

$$\sum_{k,l=0,0; y_{\sigma^{k,l}}(p,q) > \delta}^{\infty,\infty} a_{m,n,k,l} [M(y_{\sigma^{k,l}}(p,q))]^{p_{k,l}} < \max \left\{ 1, \frac{KM(2)}{\delta} \right\}^H s_{m,n}^{p,q}.$$

Thus (2.1) and RH-regularity of A grants us $\omega''(A_\sigma, p) \subset \omega''(A_\sigma, M, p)$. Following similar arguments we can prove the following: $\omega''_0(A_\sigma, p) \subset \omega''_0(A_\sigma, M, p)$, $\omega''_\infty(A_\sigma, p) \subset \omega''_\infty(A_\sigma, M, p)$. \square

Theorem 2.4. (1) If $0 < \inf p_{k,l} \leq p_{k,l} < 1$ then

$$\omega''(A_\sigma, M, p) \subset \omega''(A_\sigma, M).$$

(2) If $1 \leq p_{k,l} \leq \sup p_{k,l} < \infty$ then

$$\omega''(A_\sigma, M) \subset \omega''(A_\sigma, M, p).$$

Proof. Using the same techniques of the Theorem 2 of Savaş and Patterson [11], it is easy to prove of the theorem. \square

3. Double A -statistical

Natural density was generalized by Freeman and Sember in [14] by replacing C_1 with a nonnegative regular summability matrix $A = a_{n,k}$. Thus, if K is a subset of N then the A -density of K is given by $\delta_A(K) = \lim_n \sum_{k \in K} a_{n,k}$ if the limit exists. In this section we define the double (A_σ) -statistical convergence and establish some connections between the spaces of strong double (A_σ) -convergence sequences and the space of double (A_σ) -statistical convergence. Let $K \subset N \times N$ be a two-dimensional set of positive integers, then the A -density of K is given by

$$\delta_A^2(K) = P\text{-}\lim_{m,n} \sum_{(k,l) \in K} a_{m,n,k,l},$$

provided that the limit exists. The notion of double asymptotic density for double sequence was presented by Mursaleen and Edely in [15].

Definition 3.1. A double real numbers sequence x is said to be (A_σ) -statistically convergent on L if for every positive ϵ

$$\delta_A^2(\{(k,l) : |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon\}) = 0$$

uniformly in (p,q) .

In this case we write $x_{k,l} \rightarrow L(st''(A_\sigma))$ or $st''(A_\sigma) - \lim x = L$ and

$$st''(A_\sigma) = \{x : \exists L \in \mathbb{R}, st''(A_\sigma) - \lim x = L\}.$$

If $A = (C, 1, 1)$ then $st''(A_\sigma)$ reduces to st''_σ which is defined as follows: A double real numbers sequence x is said to be σ -statistically convergent on L , if for every positive $\epsilon > 0$ the set

$$P\text{-}\lim_{m,n} \frac{1}{mn} |\{k \leq m \text{ and } l \leq n : |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon\}| = 0$$

uniformly in (p,q) .

In this case we write $st''_\sigma - \lim x = L$. If we take

$$a_{r,s,k,l} = \begin{cases} \frac{1}{\bar{h}_{r,s}}, & \text{if } k \in I_r = (k_{r-1}, k_r] \text{ and } l \in I_s = (l_{s-1}, l_s] \\ 0 & \text{otherwise} \end{cases}$$

where the double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ and $\bar{h}_{r,s}$ are defined above. The our definition reduces to the following: A double real numbers sequence x is said to be lacunary θ -statistically convergent on L , if for every positive $\epsilon > 0$ the set

$$P\text{-}\lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon\}| = 0$$

uniformly in (p, q) . Finally, if we write

$$a_{i,j,k,l} = \begin{cases} \frac{1}{\bar{\lambda}_{i,j}}, & \text{if } k \in I_i = [i - \lambda_i + 1, i] \text{ and } l \in I_j = [j - \lambda_j + 1, j]; \\ 0, & \text{otherwise} \end{cases}$$

where $\bar{\lambda}_{i,j}$ by $\lambda_i \mu_j$. Let $\lambda = (\lambda_i)$ and $\mu = (\mu_j)$ are defined above. A double real numbers sequence x is said to be lacunary $(\bar{\lambda}, \sigma)$ -statistically convergent on L , if for every positive $\epsilon > 0$ the set

$$P\text{-}\lim_{i,j} \frac{1}{\bar{\lambda}_{i,j}} |\{k \in I_i \text{ and } l \in I_j : |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon\}| = 0$$

uniformly in (p, q) .

Theorem 3.1. *If M is an Orlicz function and $0 < h = \inf_{k,l} p_{k,l} \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$ then $\omega''(A_\sigma, M, p) \subset st''(A_\sigma)$.*

Proof. If $x \in \omega''(A_\sigma, M, p)$, then there exists $\rho > 0$ such that

$$P\text{-}\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} = 0,$$

uniformly in (p, q) . Then given $\epsilon > 0$ and let $\epsilon_1 = \frac{\epsilon}{\rho}$ we obtain the following for each (p, q)

$$\begin{aligned} & \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} = \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} \\ & + \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| < \epsilon}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} \\ & \geq \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} \\ & \geq \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} [M(\epsilon_1)]^H \\ & \geq \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} (\min\{[M(\epsilon_1)]^h, [M(\epsilon_1)]^H\}) \\ & \geq (\min\{[M(\epsilon_1)]^h, [M(\epsilon_1)]^H\}) \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} \\ & \geq (\min\{[M(\epsilon_1)]^h, [M(\epsilon_1)]^H\}) \delta_A^2(\{(k, l) : |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon\}). \end{aligned}$$

Hence $x \in st''(A_\sigma)$. \square

Theorem 3.2. *If M is a bounded Orlicz function and $0 < h = \inf_{k,l} p_{k,l} \leq p_{k,l} \leq \sup_{k,l} p_{k,l} = H < \infty$ then $st''(A_\sigma) \subset \omega''(A_\sigma, M, p)$.*

Proof. Suppose that M is bounded then there exists an integer K such that $M(x) \leq K$ for $x > 0$, and for each (p, q)

$$\begin{aligned} & \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} = \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} \\ & + \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| < \epsilon}^{\infty,\infty} a_{m,n,k,l} \left[M \left(\frac{|x_{\sigma^{k,l}(p,q)} - L|}{\rho} \right) \right]^{p_{k,l}} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon}^{\infty, \infty} a_{m,n,k,l} \max\{K^h, K^H\} + \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| < \epsilon}^{\infty, \infty} a_{m,n,k,l} \left[M\left(\frac{\epsilon}{\rho}\right) \right]^{p_{k,l}} \\
&\leq \max\{K^h, K^H\} \sum_{k,l=0,0; |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon}^{\infty, \infty} a_{m,n,k,l} + \max\{[M(\epsilon_1)]^h, [M(\epsilon_1)]^H\} \\
&\leq \delta_A^2(\{(k, l) : |x_{\sigma^{k,l}(p,q)} - L| \geq \epsilon\}) \max\{K^h, K^H\} + \max\{[M(\epsilon_1)]^h, [M(\epsilon_1)]^H\}.
\end{aligned}$$

Thus $x \in \omega''(A_\sigma, M, p)$. \square

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